Directed Isoperimetric Inequalities for Boolean Functions on the Hypergrid and an $\widetilde{O}(n\sqrt{d})$ Monotonicity Tester

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Monotonicity Testing

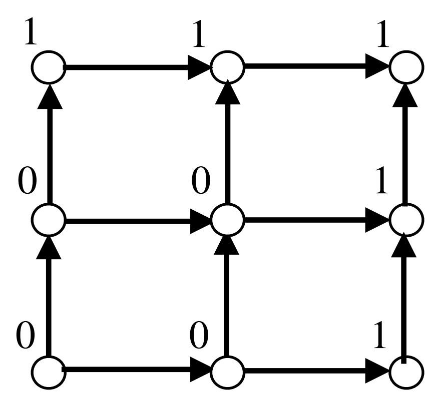
A central problem in property testing proposed by
Goldreich-Goldwasser-Lehman-Ron-Samorodnitsky 99

We consider Boolean functions over the hypergrid, $f\colon [n]^d \to \{0,1\}$

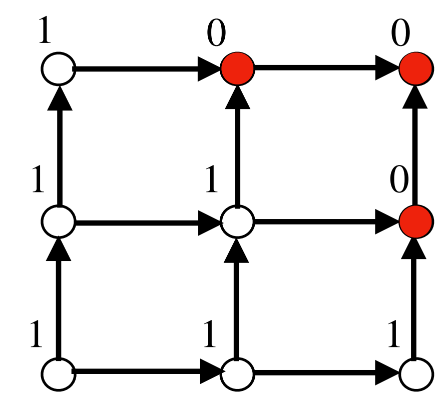
f is **monotone** if $f(x) \le f(y)$ whenever $x \prec y$

Partial order on $[n]^d$: $x \preceq y$ iff $x_i \le y_i, \forall i \in [d]$

$\varepsilon(f) = 0$



$\varepsilon(f) = 1/3$



Distance to monotonicity:

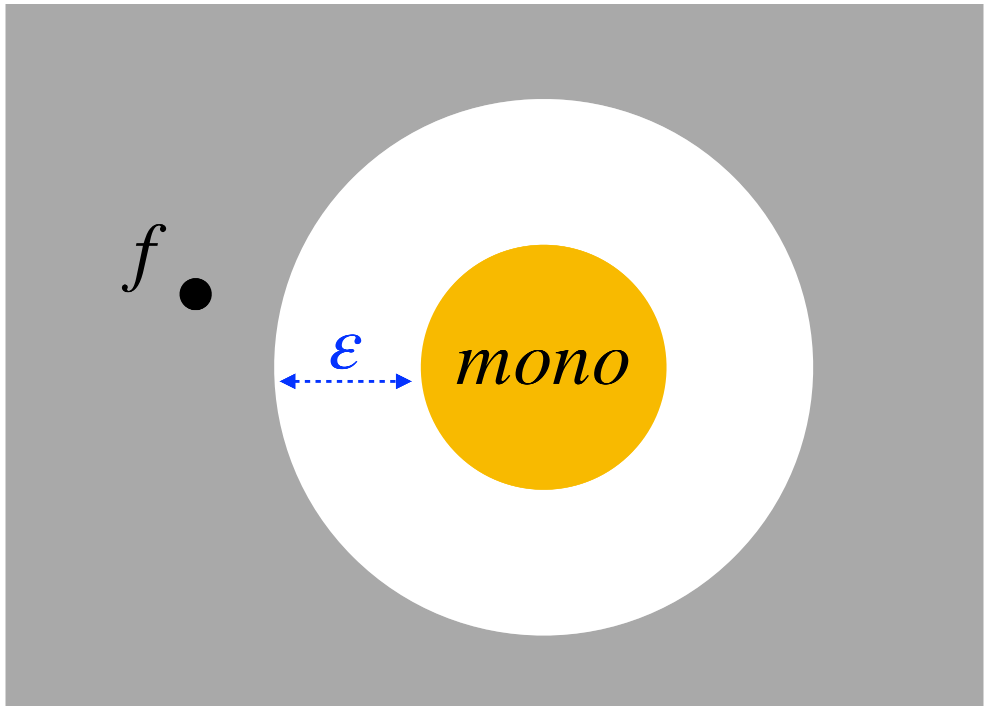
$$\varepsilon(f) = n^{-d} \cdot \min_{h \text{ monotone}} \# x : f(x) \neq h(x)$$

Monotonicity Testing

A central problem in property testing

Given $f\colon [n]^d \to \{0,1\}$ and $\varepsilon > 0$...

1. if f monotone: **accept** w.p. ~~$> 2/3$~~ 1
2. if $\varepsilon(f) > \varepsilon$: **reject** w.p. $> 2/3$



query model:

can request $f(x)$ for any element $x \in [n]^d$

non-adaptive: tester specifies all queries up front

1-sided error: always accept a monotone function

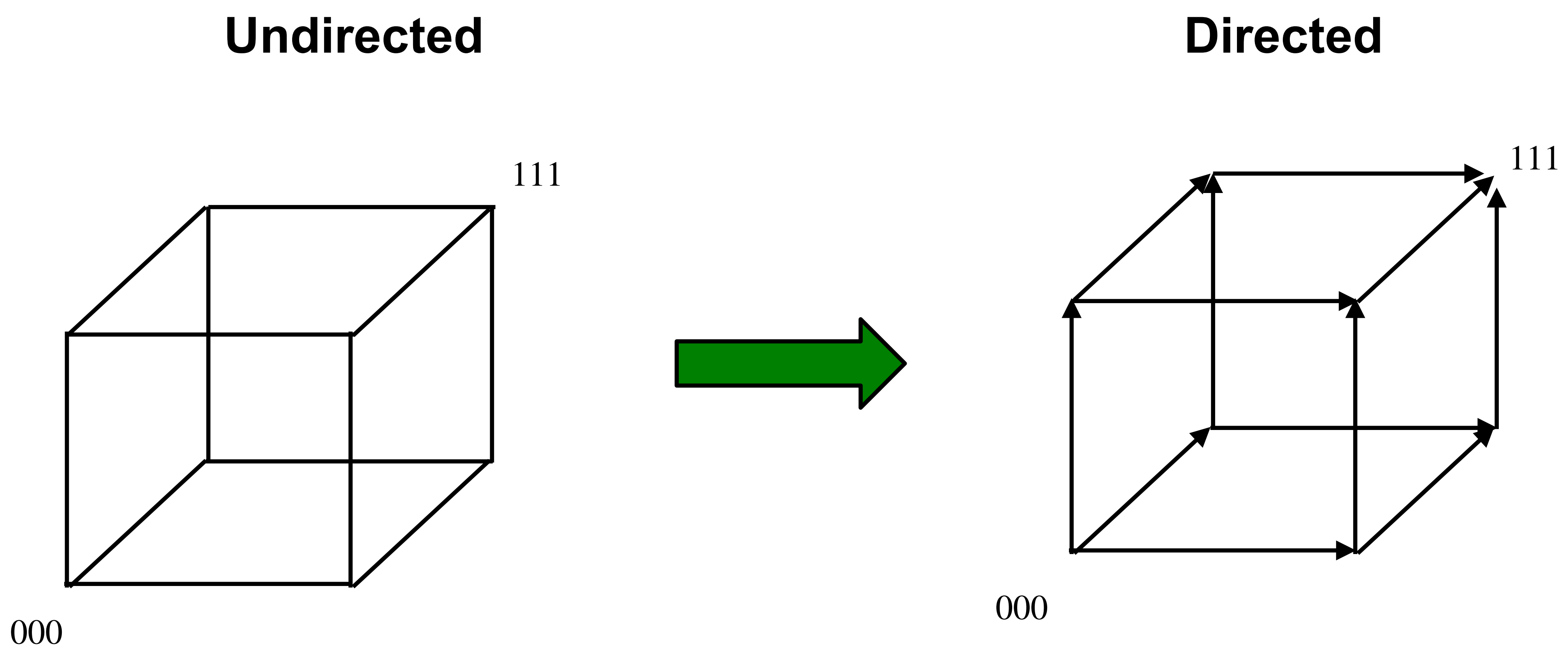
Essence of the problem:
How many queries to find a **violation of monotonicity** when f is far from any monotone function?

Testing Boolean Functions

Most well-studied setting: $n = 2$

$f: \{0,1\}^d \to \{0,1\}$

Poset = the **directed** hypercube



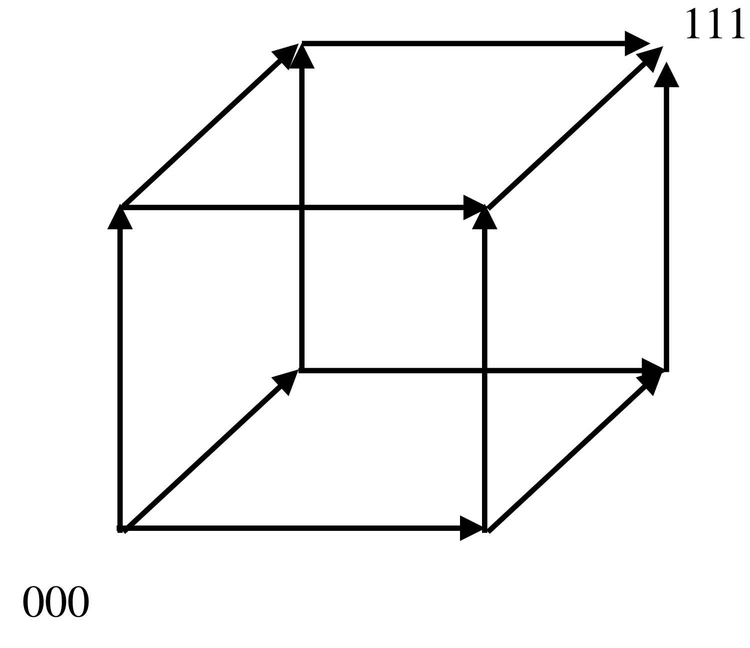
The Hypercube and Isoperimetry

(for brevity let $\varepsilon = \Omega(1)$)

- Testing results for $f: \{0,1\}^d \to \{0,1\}$
  - Goldreich, Goldwasser, Lehman, Ron, Samorodnitsky 99: $O(d)$
  - Chakrabarty Seshadhri 14: $\widetilde{O}(d^{7/8})$
  - Chen Servedio Tan 14: $\widetilde{O}(d^{5/6})$
  - Khot Minzer Safra 15: $\widetilde{O}(d^{1/2})$

key insight: connection to isoperimetry on the hypercube

Lower bound:
Chen, Waingarten, Xie 17:
- $\widetilde{\Omega}(d^{1/2})$ for non-adaptive
- $\widetilde{\Omega}(d^{1/3})$ for adaptive



Testing Results for General n

- Dodis Goldreich Lehman Raskhodnikova Ron Samorodnitsky 00: $O(d \log n)$
- Berman Raskhodnikova Yaroslavtsev 14: $O(d \log d)$
- Black Chakrabarty Seshadhri SODA 18, SODA 20: $\widetilde{O}(d^{5/6})$
- Black Chakrabarty Seshadhri STOC 23: $\widetilde{O}(nd^{1/2})$

Optimal in d Wasn't known even for $n = 3$

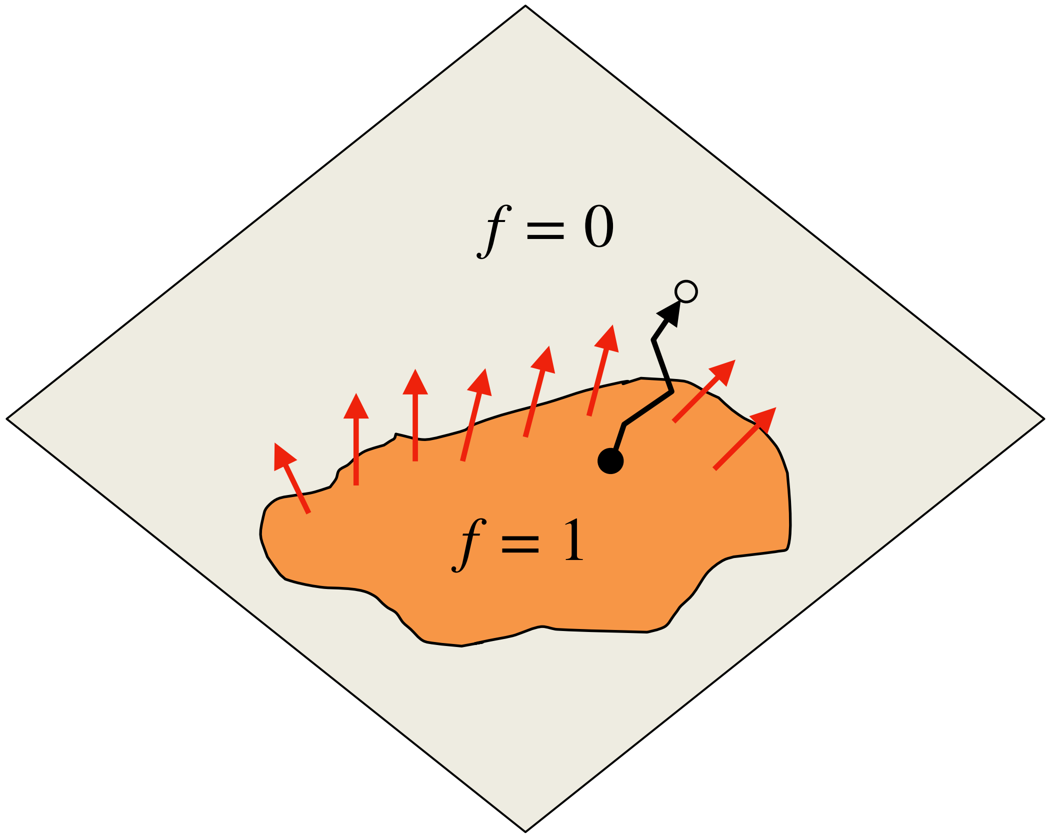
  - We extend the isoperimetric theorem by KMS [15] to all n
  - Old proofs are highly specialized to the $n = 2$ case

Parallel work:

- Braverman, Khot, Kindler, Minzer ITCS 23: $\widetilde{O}(n^3 d^{1/2})$
  - completely different techniques

Monotonicity Testing and Isoperimetry

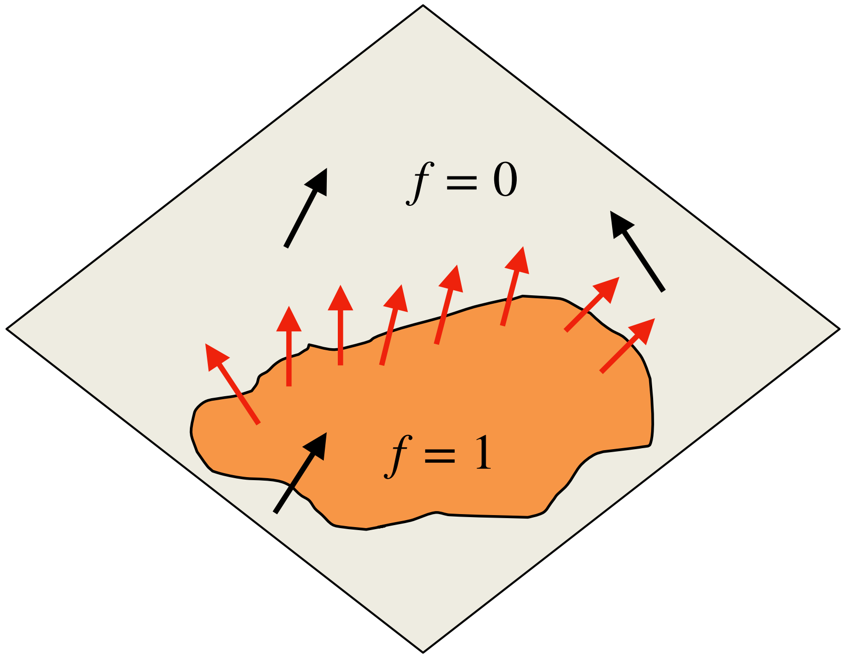
- Strategy: try to find two points $x \prec y$ where $f(x) = 1$ and $f(y) = 0$
  - If you find such a pair, then reject.
  - Otherwise, accept.
- Want to find a pair of comparable points $x \prec y$ which straddle the **upper boundary** of the set $\{x : f(x) = 1\}$



Testing and Isoperimetry over $\{0,1\}^d$

Edge tester [GGLRS '99]

- Sample an edge (x, y) in the hypercube uar.
- Reject if (x, y) is a **violation**. $(f(x) > f(y))$



What is the probability that this test finds a violation?

Negative influence

$I_f^-(x) =$ **# edges** $(x, y) : f(x) > f(y)$

$I_f^- = \mathbb{E}[I_f^-(x)]$

Total influence

$I_f(x) =$ **# edges** $(x, y) : f(x) \neq f(y)$

$I_f = \mathbb{E}[I_f(x)]$

Theorem: (GGLRS [99]) $I_f^- \geq \Omega(\varepsilon(f))$

Theorem: (Poincaré) $I_f \geq \Omega(var(f))$

$\implies$ # violated edges $\geq \Omega(\varepsilon(f)) \cdot 2^d$

Total # edges is $d \cdot 2^{d-1}$

$\implies$ Edge test succeeds with probability $\Omega(\varepsilon/d)$

$\implies$ Repeating $O(d/\varepsilon)$ times yields a tester!

Question: Is this analysis of the edge tester optimal?

Limits of the edge tester

Theorem: GGLRS [99] $I_f^- \geq \Omega(\varepsilon(f))$

Is this inequality tight? Yes.

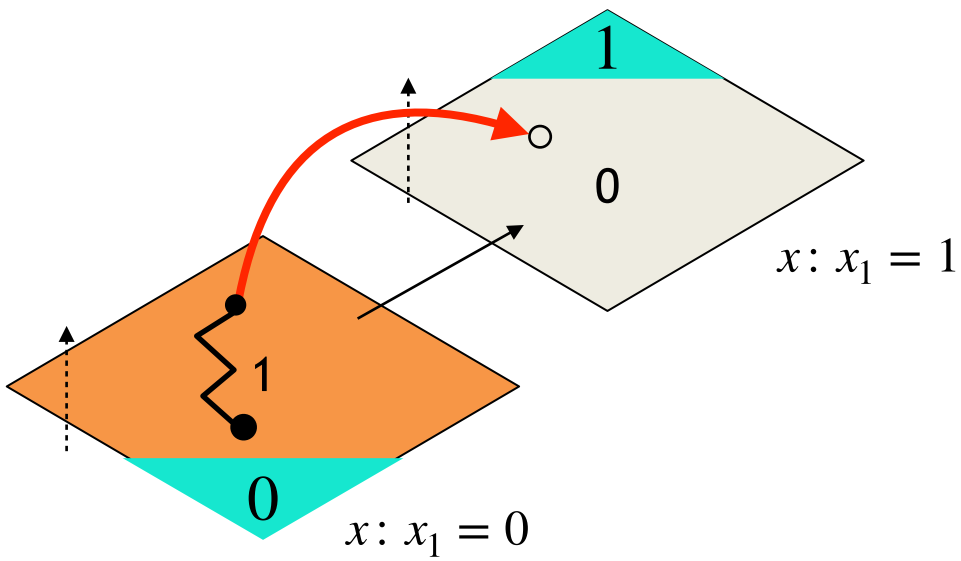
- To beat $O(d)$ requires something other than the edge tester

Path tester (informal):

- Sample x uniformly
- Obtain y by an **directed** random walk of length $\approx d^{1/2}$ from x
- Reject if $f(x) > f(y)$

- Succeeds with probability $\Omega(d^{-1/2})$ for the anti-dictator function
- Why?
  - Intuition: edge violations are **spread** amongst the vertices

anti-dictator function: $f(x) = 1 - x_1$



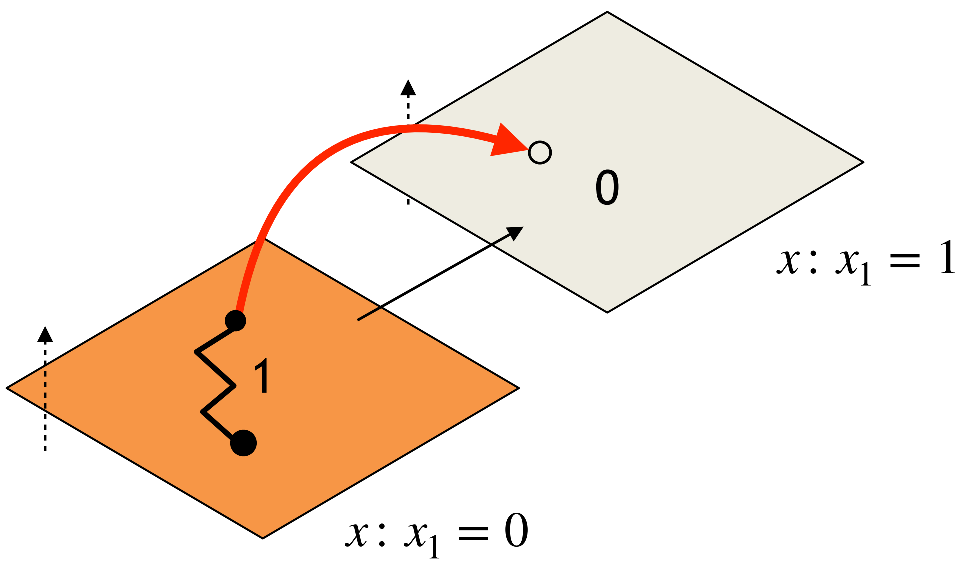
$\varepsilon(f) = 1/2$ and $I_f^- = 1/2$

Question:
Is there a more nuanced way to understand boundary that can capture this intuition?

A Nuanced Way of Capturing Boundary

<u>Example</u> 1: anti-dictator

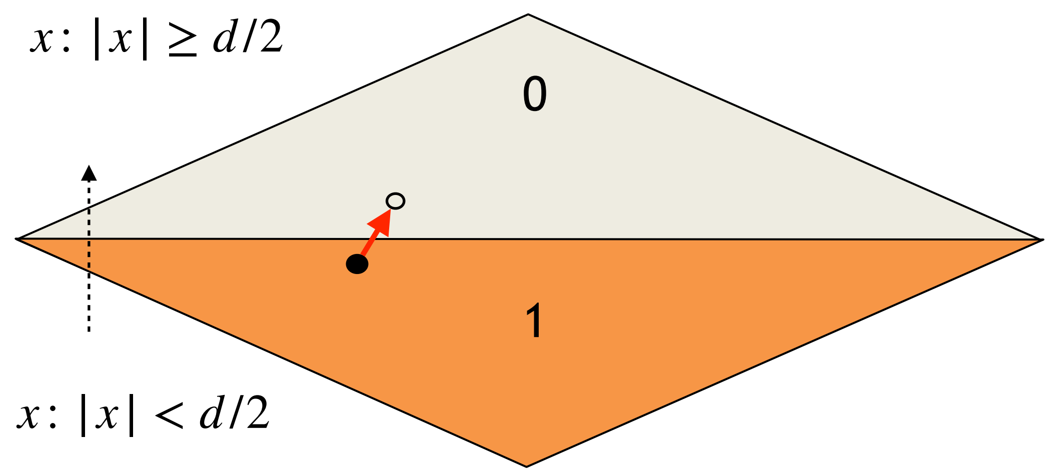
$\varepsilon(f) = 1/2$ **and** $I_f^- = \Theta(1)$



Few edge violations **spread** amongst **many** vertices

<u>Example</u> 2: anti-majority

$\varepsilon(f) = 1/2$ **and** $I_f^- = \Omega(d^{1/2})$



Many edge violations **concentrated** on **few** vertices

Both examples:
$\mathbb{E}[I_f^-(x)^{1/2}] = \Theta(1)$

Desired tradeoff
A) There are **many** edge violations or
B) All edge violations are **spread** amongst the vertices

Theorem: (KMS [15]): $\mathbb{E}[I_f^-(x)^{1/2}] = \widetilde{\Omega}(\varepsilon(f))$ $\Longrightarrow$ Can test with $\widetilde{O}(d^{1/2})$ queries

Directed Isoperimetry $\implies$ Testers

Undirected isoperimetric inequality	Directed isoperimetric inequality	Monotonicity Tester
Poincaré $I_f = \Omega(var(f))$	GGLRS [99] for $n = 2$ DGLRRS [99] for $n \geq 2$ $I_f^- = \Omega(\varepsilon(f))$	$O(d)$
Margulis [74]	CS [14] for $n = 2$ BCS [18] for $n \geq 2$	$\widetilde{O}(d^{5/6})$ CST [14], BCS [18,20]
Talagand [93] $\mathbb{E}[I_f(x)^{1/2}] = \Omega(var(f))$	KMS [15] for $n = 2$ $\mathbb{E}[I_f^-(x)^{1/2}] = \Omega(\frac{\varepsilon(f)}{\log d})$	$\widetilde{O}(\sqrt{d})$

$\log d$ removed by
by Pallavoor, Raskhodnikova, Waingarten [20]

Our contribution:

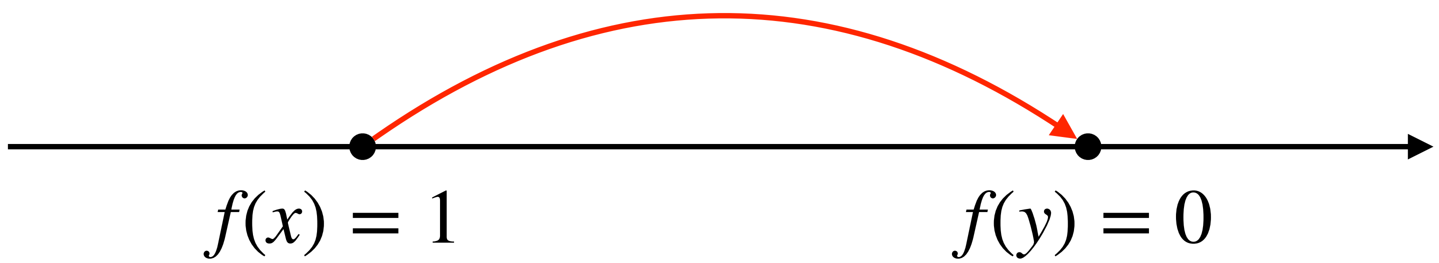
We generalize Khot-Minzer-Safra's inequality to all $n \geq 2$ $\implies$ Obtain a $\widetilde{O}(n\sqrt{d})$ query monotonicity tester

Our Isoperimetric Theorem for Hypergrids

New notion of boundary

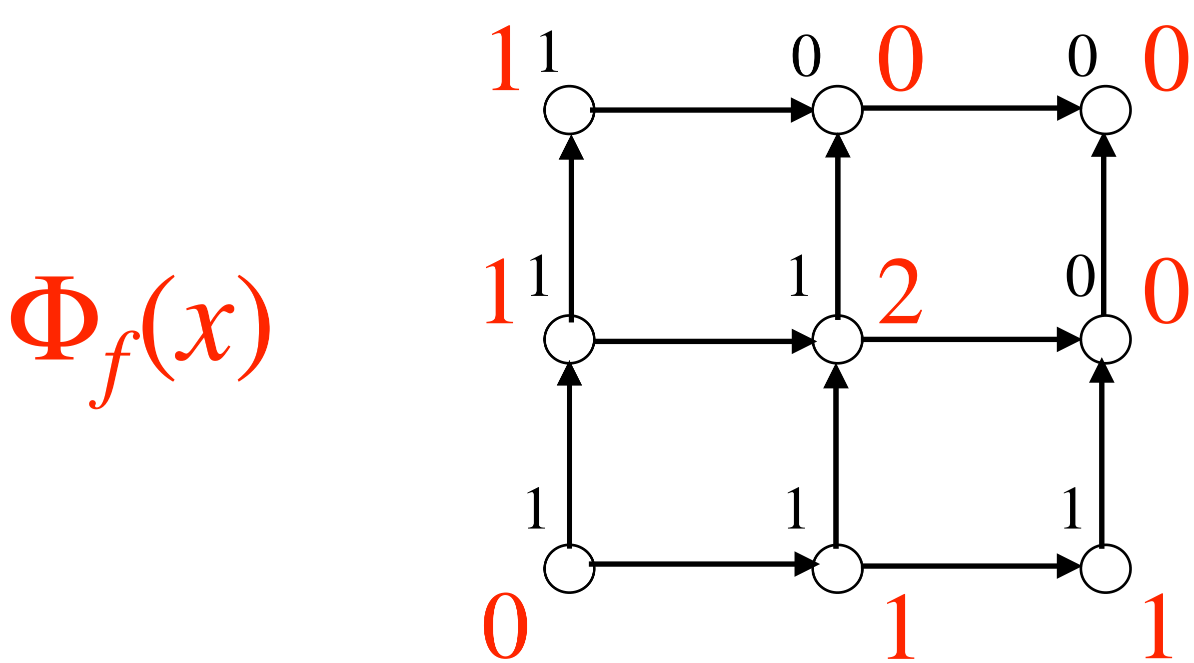
Thresholded Influence

<u>Terminology:</u> Given $f\colon [n]^d \to \{0,1\}$ an i-**violation** is a pair (x, y) which differ only in coordinate i and violate monotonicity of f.



<u>Defn:</u> [Thresholded Influence] Given $f\colon [n]^d \to \{0,1\}$ and $x \in [n]^d$,

$$\Phi_f(x) = \#i \in [d] : x \text{ is the lower endpoint of an } i\text{-violation}$$



Directed Talagrand on $[n]^d$

Defn: [Thresholded Influence] Given $f\colon\ [n]^d \to \{0,1\}$ and $x \in [n]^d$,

$$\Phi_f(x) = \#i \in [d] : x \text{ is the lower endpoint of an } i\text{-violation}$$

Theorem: (**B**CS [23]) For any $f\colon\ [n]^d \to \{0,1\}$,

$$\mathbb{E}[\Phi_f(x)^{1/2}] = \Omega\left(\frac{\varepsilon(f)}{\log n}\right)$$

- When $n = 2,\ \Phi_f(x) = I_f^-(x)$
- Generalizes the inequality of KMS [15] to any n
- BKKM [23] prove the same inequality, but with $poly(n)$ in the denominator of the RHS

Robust Directed Talagrand on $[n]^d$

Let E denote the set of pairs (x, y) which differ in exactly one coordinate

Defn: [Colorful Influence] Given $f: [n]^d \to \{0,1\}$, $\chi: E \to \{0,1\}$, and $x \in [n]^d$,

$$\Phi_{f,\chi}(x) = \#i \in [d]: x \text{ participates in an } i\text{-violation } (x, y) \text{ where } \chi(x, y) = f(x).$$

Theorem: (BCS [23]) For any $f: [n]^d \to \{0,1\}$ and $\chi: E \to \{0,1\}$

$$\mathbb{E}[\Phi_{f,\chi}(x)^{1/2}] = \Omega\left(\frac{\varepsilon(f)}{\log n}\right)$$

- [KMS15] proved this inequality for $n = 2$
- Robustness makes the proofs **much** more challenging

Theorem: (BCS [23]) There is a $\widetilde{O}(n\sqrt{d})$ query monotonicity tester.

- Optimal dependence on d comes from our isoperimetric inequality
- Suboptimal dependence on n: underlying graph has degree n in each dimension

Proof ideas

Theorem: (BCS [23]) For any $f\colon [n]^d \to \{0,1\}$,

$$\mathbb{E}\left[\Phi_f(x)^{1/2}\right] = \Omega\left(\frac{\varepsilon(f)}{\log n}\right)$$

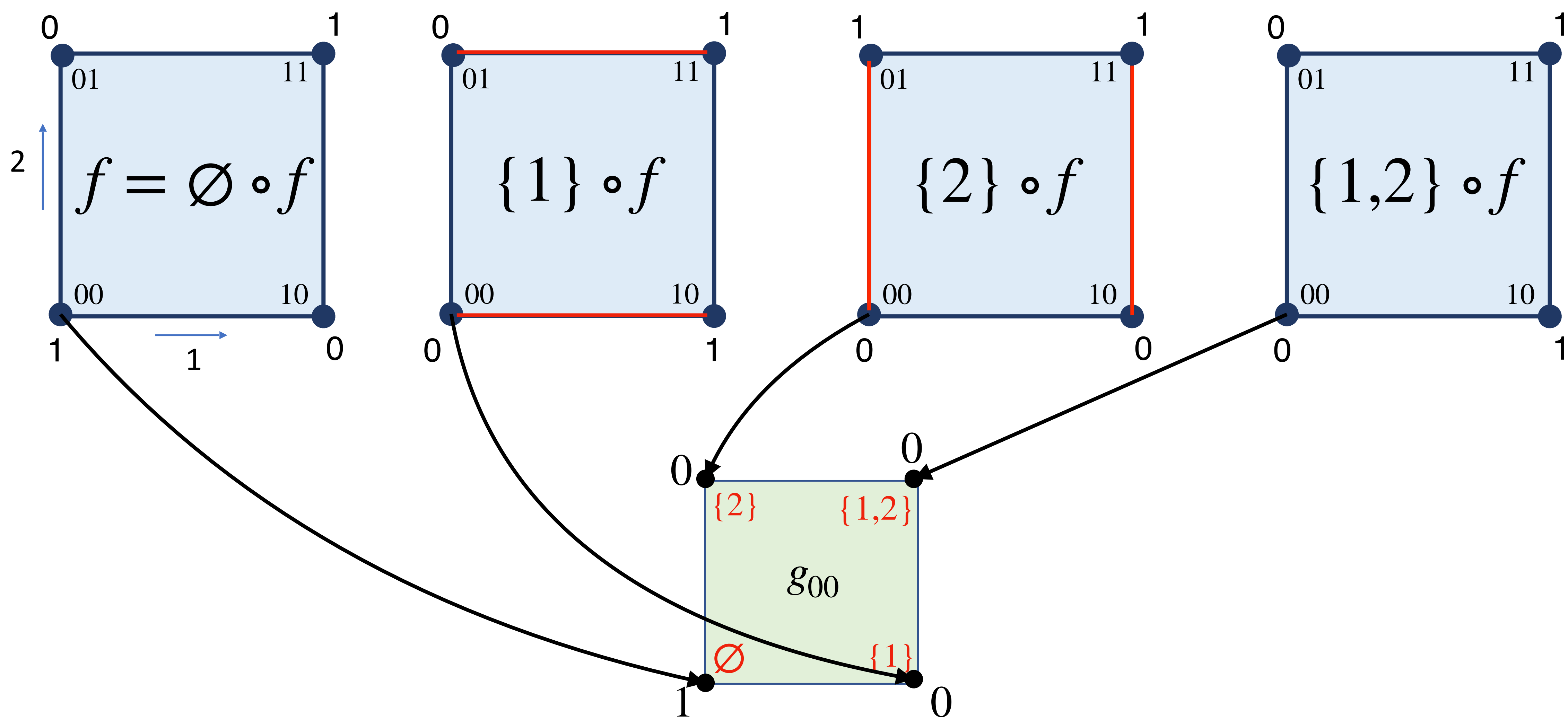
Idea 1: tracking the effects of sorting

Defn: [Sort operator] Given $f\colon [n]^d \to \{0,1\}$ and $S \subseteq [d]$:

- $S \circ f =$ function obtained by sorting f on each coordinate in S (in increasing order)

Defn: [Tracker functions] Given $f\colon [n]^d \to \{0,1\}$ and $\mathbf{x} \in [n]^d$ define

$$g_{\mathbf{x}}\colon 2^{[d]} \to \{0,1\} \text{ as } g_{\mathbf{x}}(S) = S \circ f(\mathbf{x}).$$



Tracking the effects of sorting

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- $S \circ f =$ function obtained by sorting f on each coordinate in S (in increasing order)

Defn: [Tracker functions] Given $f\colon [n]^d \to \{0,1\}$ and $x \in [n]^d$ define

$g_x\colon 2^{[d]} \to \{0,1\}$ as $g_x(S) = S \circ f(x)$.

Main inequality

$$\mathbb{E}[\Phi_f(\mathbf{x})^{1/2}] \geq \mathbb{E}_{\mathbf{x}\in[n]^d}\mathbb{E}_{S\subseteq[d]}[I_{g_{\mathbf{x}}}(S)^{1/2}]$$

Directed boundary of f

Average **undirected** boundary of $g_{\mathbf{x}}$'s

Can leverage undirected Talagrand on the hypercube to bound RHS

Proof big picture

1. Connecting to the tracker functions

*If f is semi-sorted

(by Talagrand [93])

$$\mathbb{E}[\Phi_f(\mathbf{x})^{1/2}] \geq \mathbb{E}_{\mathbf{x}}\mathbb{E}_S[I_{g_{\mathbf{x}}}(S)^{1/2}] \geq C \cdot \mathbb{E}_x[var(g_x)]$$

Proof strategy:

- A "hybrid argument" which transforms the LHS into the RHS in d steps
- Inspired by the split operator of KMS [15]
    - Unclear how to generalize the split operator to hypergrids
    - Our key tool: vector **majorization**

2. Reduction to semi-sorted functions

Defn: $f\colon [n]^d \rightarrow \{0,1\}$ is **semi-sorted** if f is monotone in every orthant.

Remark: All functions $f\colon \{0,1\}^d \rightarrow \{0,1\}$ are semi-sorted.

3. Connecting $\mathbb{E}_x[var(g_{\mathbf{x}})]$ back to $\varepsilon(f)$

- Proof is similar to arguments in KMS [15] and PRW [20]

Summary

- We generalize the directed Talagrand inequality of Khot-Minzer-Safra to the hypergrid $[n]^d$
- Obtain a monotonicity tester making $\widetilde{O}(n\sqrt{d})$ queries
  - First achieving $\sqrt{d}$ for any $n > 2$ (alongside parallel work of BKKM [23])

Open question 1:
Is there a $\widetilde{O}(\sqrt{d})$ tester with no dependence on n?

- Yes! (nearly) Our recent follow up work achieves $d^{1/2+o(1)}$ queries
  - Analysis relies on our directed Talagrand inequality over $[n]^d$
  - Resolves non-adaptive Boolean monotonicity testing (nearly) for all n

Open question 2:
Can we achieve $o(\sqrt{d})$ queries with **adaptive** testers?

- Current lower bound is $\widetilde{\Omega}(d^{1/3})$ by Chen-Waingarten-Xie [17]

Thank you!